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ON OBTAINING A SOLUTION OF THE EQUATION FOR AN AIRFOIL

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A number of artificial procedures is used to reduce the basic, two-dimensional singular integral equation of the linear theory of thin airfoil of arbitrary aspect ratio, arbitrary form in the plane, and with the angle of attack varying along the wing span, to a form allowing the separation of variables. It is shown that the solution of the resulting equation can be reduced to solving certain homogeneous Riemann's boundary value problems admitting a solution by means of the Cauchy type integrals [1].

The basic equation in question which has been derived using the framework of the theory of acceleration potential has not, so far, been solved in the closed, analytic form. It has been the subject of numerous investigations and publications aiming mainly to obtain more or less correct approximations to the kernel with the view of obtaining the homogeneous equations [2].

1. The basic equation of the theory of thin airfoil S moving in a straight line with constant velocity u through an unbounded medium, is written in the form [2]

$$\frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{A^\circ(x_0, y_0)}{y-y_0}\left[1-\frac{R(x_0, y_0, x, y)}{x-x_0}\right]dx_0\,dy_0 = -V(y)+C(x) \qquad (1.1)$$

$$A^\circ(x, y)=\frac{A(x, y)}{2\lambda(y)},\quad \lambda(y)=\frac{b}{a(y)},\quad V(y)=u\int V_*(y)\,dy$$

$$R(x_0, y_0, x, y)=[(x-x_0)^2+\lambda^2(y_0)(y-y_0)^2]^{1/2}$$

$$x, y, x_0, y_0\in S_*$$

Here $A(x, y)$ is the solution sought, $\lambda(y)$ is the wing aspect ratio, b its half-span, $a(y)$ is the half-chord in the cross section y, $V_*(y)$ is the component of the stream velocity normal to S, $C(x)$ is a function of x not known in advance and S_* is the projection of S onto the flight plane.

Equation (1.1) describes a class of thin wings twisted hydrodynamically along the span, with a symmetric profile and of arbitrary form in the plane. Taking into account the fact that

$$\frac{R(x_0, y_0, x, y)}{(y-y_0)(x-x_0)}=R^{-1}(x_0, y_0, x, y)\left[\frac{x-x_0}{y-y_0}+\lambda^2(y_0)\frac{y-y_0}{x-x_0}\right]$$

we can write the equation (1.1) in the form

$$\frac{1}{2\pi}\int_{-1}^{1}[\varphi_1(y_0)-\varphi_2(x, y_0, y)]\frac{dy_0}{y-y_0} = \tag{1.2}$$

$$\frac{1}{2\pi}\int_{-1}^{1}\varphi_3(x, y_0, y)(y-y_0)\,dy_0 - V(y)+C(x)$$

$$\varphi_1(y_0)=\int_{-1}^{1}A^\circ(x_0, y_0)\,dx_0, \quad \varphi_2(x, y_0, y)=\int_{-1}^{1}A^\circ(x_0, y_0)\frac{(x-x_0)\,dx_0}{R(x_0, y_0, x, y)} \tag{1.3}$$

$$\varphi_3(x, y_0, y)=\int_{-1}^{1}\frac{A^\circ(x_0, y_0)\lambda^2(y_0)}{R(x_0, y_0, x, y)}\frac{dx_0}{x-x_0}$$

Let us assume that the function $A^\circ(x, y)$ satisfies the Hölder condition [1] in both variables. Then the functions φ_1 and φ_2 will clearly satisfy this condition as well, in all corresponding variables. This enables us, irrespective of the fact that the function φ_2 depends on the parameters x and y, to use the known formulas for inverting the Cauchy type integrals in the corresponding class of functions [3]. Since the pressure jump at the wing, proportional to the function A, should vanish at the side edges, we shall invert the singular integral appearing in the left hand side of (1.2), in the class of functions bounded at the points $y=\pm 1$. This yields

$$\varphi_1(y)-\varphi_2(x, y_0, y) = -\frac{\sqrt{1-y^2}}{\pi}\int_{-1}^{1}\frac{2V(y_0)-2C(x)-\varphi_4(x, y_0)}{\sqrt{1-y_0^2}(y-y_0)}\,dy_0 \tag{1.4}$$

$$\varphi_4(x, y)=\frac{1}{\pi}\int_{-1}^{1}(y-y_0)\varphi_3(x, y_0, y)\,dy_0 \tag{1.5}$$

under the condition

$$\int_{-1}^{1}\frac{2V(y_0)-2C(x)-\varphi_4(x, y_0)}{\sqrt{1-y_0^2}}=0 \tag{1.6}$$

satisfied by the suitable choice of $C(x)$.

We note that the result (1.4) can be obtained without resorting to the readily available inversion formulas, but deriving them from the solution of the corresponding Riemann boundary value problem [1]. In this case the equation (1.6) will serve as the condition of solvability of the Riemann problem in the class of bounded functions.

Using (1.3) we can show that the left-hand side of (1.4) has the form

$$\int_{-1}^{1}A^\circ(x_0, y)\,dx_0-\int_{-1}^{1}A^\circ(x_0, y)\,\mathrm{sign}\,(x-x_0)\,dx_0 = 2\int_{x}^{1}A^\circ(x_0, y)\,dx_0$$

and for any function $C(x)$ we have, in accordance with (1.4),

$$\int_{x}^{1}A^\circ(x_0, y)\,dx_0=\frac{\sqrt{1-y^2}}{2\pi}\int_{-1}^{1}\frac{\varphi_4(x, y_0)-2V(y_0)}{\sqrt{1-y_0^2}(y-y_0)}\,dy_0$$

The independence of the variables x and y means that, having rewritten the above equation with (1.5) taken into account, we can change the order of integration, i.e. we can write

$$\int_{-1}^{1} \frac{dx_0}{x-x_0} \frac{\sqrt{1-y^2}}{\pi^2} \int_{-1}^{1}\int_{-1}^{1} \frac{A^\circ(x_0, y_1)\lambda^2(y_1)}{R(x_0, y_1, x, y_0)} \frac{(y_0-y_1)\,dy_1 dy_0}{\sqrt{1-y_0^2}\,(y-y_0)} = B(x, y) \qquad (1.7)$$

$$\frac{1}{2} B(x, y) = \int_x^1 A^\circ(x_0, y)\,dx_0 + \frac{\sqrt{1-y^2}}{\pi} \int_{-1}^{1} \frac{V(y_0)\,dy_0}{\sqrt{1-y_0^2}\,(y-y_0)}$$

Inverting now the integral appearing in the left-hand side of the first of the above equations in the class of functions unbounded on the leading edge and bounded on the trailing edge of the wing (Joukowski —Chaplygin postulate [2]), we obtain

$$\sqrt{1-y^2} \int_{-1}^{1}\int_{-1}^{1} A^\circ(x, y_1)\,\lambda(y_1)\,\mathrm{sign}\,(y_0-y_1) \frac{dy_1 dy_0}{\sqrt{1-y_0^2}\,(y-y_0)} =$$

$$\sqrt{\frac{1+x}{1-x}} \int_{-1}^{1} \sqrt{\frac{1-x_0}{1+x_0}}\, B(x_0, y) \frac{dx_0}{x_0-x}$$

Applying to this equation the operators

$$M(u) = \int_{-1}^{1} u \frac{dy_0}{y_0-y} \quad \text{and} \quad L(v) = \int_{-1}^{1} v \frac{dx_0}{x_0-x}$$

we express it in the following form:

$$\frac{1}{2} \int_{-1}^{1} \frac{dx_0}{x_0-x} \left[D(x_0) + \int_{-1}^{1} A(x_0, y_0)\,\mathrm{sign}\,(y-y_0)\,dy_0 \right] = \qquad (1.8)$$

$$\int_{-1}^{1} B(x, y_0) \frac{dy_0}{y_0-y} \qquad (A = 2\lambda A^\circ)$$

The function $D(x)$ generated by the operator M, can be expressed in terms of $C(x)$; this is however immaterial in what follows. Indeed, since we have

$$\int_{-1}^{1} B(x, y_0) \frac{dy_0}{y_0-y} = \pi i B(x, y) - \int_{-1}^{1} \frac{\partial B(x, y_0)}{\partial y_0} \ln(y_0-y)\,dy_0$$

$$(B(x, \pm 1) = 0)$$

$$\frac{\partial}{\partial y} \int_{-1}^{1} \frac{\partial B(x, y_0)}{dy_0} \ln(y_0-y)\,dy_0 = -\int_{-1}^{1} \frac{\partial B(x, y_0)}{\partial y_0} \frac{dy_0}{y_0-y} +$$

$$\pi i \frac{\partial B(x, y)}{\partial y}$$

differentiation of (1.8) with respect to y yields

$$\int_{-1}^{1} \frac{\partial B(x, y_0)}{\partial y_0} \frac{dy_0}{y_0 - y} = \frac{1}{2} \int_{-1}^{1} A(x_0, y) \frac{dx_0}{x_0 - x}$$

From the second formula of (1.7) it follows

$$A(x, y) = -\lambda(y) \frac{\partial B(x, y)}{\partial x}$$

and we can therefore write the previous equation in the form

$$\frac{1}{\lambda(y)} \int_{-1}^{1} \frac{\partial B(x, y_0)}{\partial y_0} \frac{dy_0}{y_0 - y} = -\frac{1}{2} \int_{-1}^{1} \frac{\partial B(x_0, y)}{\partial x_0} \frac{dx_0}{x_0 - x}$$

Setting now $B(x, y) = X(x) Y(y)$ and separating the variables, we finally obtain

$$\int_{-1}^{1} \frac{dY(y_0)}{dy_0} \frac{dy_0}{y_0 - y} = \mu\lambda(y) Y(y) \tag{1.9}$$

$$\int_{-1}^{1} \frac{dX(x_0)}{dx_0} \frac{dx_0}{x_0 - x} = -2\mu X(x)$$

where μ is the constant of separation.

The equations obtained, formally coincide with the Prandtl equation of the theory of one-dimensional (line) airfoil [4].

Equations (1.9) can be solved using the method described in Sect. 2.

2. The function $Y(y)$ which is to be determined, is a solution of the following, one-dimensional singular integro-differential equation:

$$\frac{1}{\pi i} \int_{-1}^{1} \frac{dY(y_0)}{dy_0} \frac{dy_0}{y_0 - y} = \lambda(y) Y(y) \tag{2.1}$$

in which $\lambda = \lambda(y)$ is a given, smooth real function of its argument. Introducing the auxilliary function

$$\Phi(z) = \frac{1}{\pi i} \int_{-1}^{1} \frac{dY}{dy_0} \frac{dy_0}{y_0 - z}, \quad \operatorname{Re} z = y$$

we can reduce the solution of (2.1) to solving the following boundary value problem with the boundary condition containing the derivatives

$$\frac{dF^+(y)}{dy} + \frac{dF^-(y)}{dy} - \lambda(y)[F^+(y) - F^-(y)] = 0 \quad (y \in (-1, 1)) \tag{2.2}$$

$$\left(F^\pm(y) = \int \Phi^\pm(y)\, dy + \text{const}, \ Y(y) = F^+(y) - F^-(y)\right)$$

Here $F^\pm(y)$ and $\Phi^\pm(y)$ are the limiting values of the functions $F(z)$ and $\Phi(z)$ attained by approaching the segment (-1, 1) from different directions.

Introducing new functions $Q^\pm(y)$ by means of the relations

$$\int Q^\pm(y)\, dy = \exp\left(\mp \int \lambda(y)\, dy\right) F^\pm(y)$$

we can write the boundary condition (2. 2) in the form

$$Q^{+}(y) = -\exp\left(2\int \lambda(y)\,dy\right) Q^{-}(y) \qquad (2.3)$$

characteristic for the homogeneous Riemann boundary value problem [1].

The coefficient of the problem (2. 3) has no singularities and its exponential index in always positive, consequently the problem has a solution which can be expressed in terms of the Cauchy type integrals [1].

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